Int J Theor Phys (2009) 48: 2633–2643
DOI 10.1007/s10773-009-0051-9

Approximate Analytical Solutions of the Dirac Equation with the Hyperbolic Potential in the Presence of the Spin Symmetry and Pseudo-spin Symmetry

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Received: 23 March 2009 / Accepted: 22 May 2009 / Published online: 2 June 2009

Abstract By employing a new improved approximation scheme to deal with the centrifugal term and pseudo-centrifugal term, we solve approximately the Dirac equation with the hyperbolic potential for the arbitrary spin-orbit quantum number κ. Under the condition of the spin and pseudospin symmetry, the bound state energy eigenvalues and the associated two-component spinors of the Dirac particle are obtained approximately by using the basic concept of the supersymmetric shape invariance formalism and the function analysis method.

Keywords Dirac equation · Hyperbolic potential · Spin and pseudospin symmetry

1 Introduction

The hyperbolic potential model introduced by Schiöberg [1] is given by

$$V(r) = D(1 - \sigma \coth \alpha r)^2, \tag{1}$$

where three adjustable parameters D, σ, and α are positive, $\sigma < 1$, and the parameter α is related to the range of the potential and has dimension of length. The hyperbolic potential function (1) has the minimum value $V(r_0) = 0$ at $r_0 = \frac{1}{\alpha} \arg\tanh \sigma$, where r_0 is the equilibrium distance (bond length) between nuclei. The potential (1) goes to infinity at the point

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of $r = 0$, and approaches V_0 exponentially for large r. For the experimental Rydberg-Klein-Rees (RKR) potential curves and the rotational-vibrating levels of some diatomic molecules, the potential (1) is more accurate than the well-known Morse potential [1]. The potential (1) is closely related to the Morse and the Coulomb potential functions [2, 3]. The potential (1) is also the special case of the multi-parameter exponential-type potential model [4, 5]. Lu et al. [2, 3] studied bound state solutions of the Schrödinger equations with the hyperbolic potential (1) by using the Pekeris-like approximation scheme to deal with the centrifugal term. By employing the same approximation scheme to deal with the centrifugal term, some authors [6, 7] have studied approximately the bound state solutions of the Dirac equation and Klein-Gordon equation with the hyperbolic potential (1). Dong et al. [8, 9] also studied the bound state solutions of the Schrödinger equation with the potential (1) by employing the conventional approximation scheme suggested by Greene and Aldrich [10] to deal with the centrifugal term. Some authors have also used the conventional approximation scheme [10] to deal with the pseudo-centrifugal term, and studied the pseudospin symmetric solutions of the Dirac equations with the Eckart potential [11–13], Pöschl-Teller potential [14] and Manning-Rosen potential [15]. The conventional approximation to the centrifugal term is a good approximation for the short-range potential, small α, but it is not a good approximation for the long-range potential, large α [8, 9]. Recently, Wei et al. [16, 17] have also studied the bound and scattering state solutions of the arbitrary l-wave Klein-Gordon equations with the Eckart potential and Manning-Rosen potential by using an alternative approximation scheme proposed by one of the present authors and collaborators [18] to deal with the centrifugal term.

Within the framework of the Dirac equation, the spin symmetry occurs when the difference potential between the vector potential $V(r)$ and scalar potential $S(r)$ is a constant (i.e., $V(r) - S(r) = \text{const.}$), and the pseudospin symmetry occurs when the sum potential of the vector potential $V(r)$ and scalar potential $S(r)$ is a constant (i.e., $V(r) + S(r) = \text{const.}$) [19–21]. The spin symmetry is relevant for mesons [22]. The pseudospin symmetry in nuclear theory refers to quasi-degeneracy of single-nucleon doublets and can be characterized with the non-relativistic quantum numbers $(n, l, j = l + 1/2)$ and $(n - 1, l + 2, j = l + 3/2)$, where n, l, and j are the single-nucleon radial, orbital, and total angular momentum quantum numbers, respectively [23, 24]. The pseudospin symmetry can be used to explain features of deformed nuclei [25], superdeformation [26] and to establish an effective shell-model coupling scheme [27]. Within the framework of the relativistic Hartree approach, Marcos et al. [28] investigated the reliability of the pseudospin symmetry in atomic nuclei, and found that the nuclear surface strongly increases the effect of pseudospin-orbital potential. In real nuclei, the pseudospin symmetry is only an approximation, and the quality of the pseudopsin symmetry approximation depends on the competition between the pseudo-centrifugal potential and the pseudospin orbital potential [29]. Alhaidari et al. [30] also studied in detail the physical interpretation on the three-dimensional Dirac equation in the cases of the spin symmetry limitation $(V(r) - S(r) = 0)$ and pseudospin symmetry limitation $(V(r) + S(r) = 0)$.

Motivated by the success in obtaining approximately the bound state solutions of the Schrödinger equations with the Hulthén potential [31], Manning-Rosen potential [32] and Eckart potential [33], and in investigating the bound state solutions of the Dirac equations with the generalized Pöschl-Teller potential [34] and Manning-Rosen potential [35], we attempt to study approximately the Dirac equation with the hyperbolic potential (1) for the arbitrary spin-orbit quantum number κ by using a new improved approximation scheme [31–35] to deal with the centrifugal term and pseudo-centrifugal term. Under the conditions of the spin symmetry and pseudospin symmetry, we study the bound state energy equation and corresponding spinor wave functions in terms of the basic concept of the supersymmetric shape invariance formalism and the function analysis method.

2 Basic Equation for the F and G Component of the Dirac Spinor

In spherical coordinates, the Dirac equation with both scalar potential $S(r)$ and vector potential $V(r)$ reads ($\hbar = c = 1$)

$$\{\boldsymbol{\alpha} \bullet \mathbf{p} + \beta[M + S(r)]\}\Psi(r,\theta,\phi) = [E - V(r)]\Psi(r,\theta,\phi), \tag{2}$$

where E is the relativistic energy of the system, M is the mass of a particle, $\mathbf{p}$ is the momentum operator, $\mathbf{p} = -i\nabla$, $\boldsymbol{\alpha}$ and β are the Dirac matrices. For a particle in a spherical field, the spin-orbit matrix operator $K = -\beta(\boldsymbol{\sigma}\cdot\mathbf{L} + 1)$ and total angular momentum operator $\mathbf{J}$ commute with the Dirac Hamiltonian, where $\boldsymbol{\sigma}$ and $\mathbf{L}$ are the Pauli matrix and orbital angular momentum, respectively. The eigenvalues of K are $\kappa = -(j + 1/2)$ for aligned spin ($s_{1/2}, p_{3/2}$, etc.) and $\kappa = (j + 1/2)$ for unaligned spin ($p_{1/2}, d_{3/2}$, etc.). One can take (H, K, J^2, J_z) as the complete set of the conservative quantities. By using the radial quantum number n and spin-orbit quantum number κ, the spinor wave functions can be classified and are given by

$$\Psi_{n\kappa}(r,\theta,\phi) = \frac{1}{r}\begin{bmatrix} F_{n\kappa}(r)Y_{jm}^{l}(\theta,\phi) \\ iG_{n\kappa}(r)Y_{jm}^{\tilde{l}}(\theta,\phi) \end{bmatrix}, \tag{3}$$

where $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are the radial wave functions of the upper and lower components, respectively, $Y_{jm}^{l}(\theta,\phi)$ and $Y_{jm}^{\tilde{l}}(\theta,\phi)$ are the spherical harmonic functions which are coupled to the angular momentum j and its projection m on the third axis. The orbital angular momentum quantum numbers l and $\tilde{l}$ refer to the upper and lower components, respectively. The quasi-degenerate doublets can be expressed by using the pseudospin angular momentum $\tilde{s} = 1/2$ and pseudo-orbital angular momentum $\tilde{l}$, which is defined as $\tilde{l} = l + 1$ for aligned spin $j = \tilde{l} - 1/2$ and $\tilde{l} = l - 1$ for unaligned spin $j = \tilde{l} + 1/2$. For example, $(3s_{1/2}, 2d_{3/2})$ can be expressed as pseudospin doublets $(2\tilde{p}_{1/2}, 2\tilde{p}_{3/2})$. Substituting (3) into (2) leads us to write the radial Dirac equations satisfied by the upper component $F_{n\kappa}(r)$ and lower component $G_{n\kappa}(r)$ as follows

$$\left(\frac{d}{dr} + \frac{\kappa}{r}\right)F_{n\kappa}(r) = [M + E_{n\kappa} + S(r) - V(r)]G_{n\kappa}(r), \tag{4}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r}\right)G_{n\kappa}(r) = [M - E_{n\kappa} + S(r) + V(r)]F_{n\kappa}(r). \tag{5}$$

By eliminating $G_{n\kappa}(r)$ in (4) and $F_{n\kappa}(r)$ in (5), we get the following two second-order differential equations satisfied by the upper and lower components of the Dirac spinor, respectively,

$$\left(-\frac{d^2}{dr^2} + \frac{\kappa(\kappa+1)}{r^2} + (M + E_{n\kappa} - \Delta)(M - E_{n\kappa} + \Sigma) - \frac{\frac{d\Delta}{dr}(\frac{d}{dr} + \frac{\kappa}{r})}{M + E_{n\kappa} - \Delta}\right)F_{n\kappa}(r) = 0, \tag{6}$$

$$\left(-\frac{d^2}{dr^2} + \frac{\kappa(\kappa-1)}{r^2} + (M + E_{n\kappa} - \Delta)(M - E_{n\kappa} + \Sigma) + \frac{\frac{d\Sigma}{dr}(\frac{d}{dr} - \frac{\kappa}{r})}{M - E_{n\kappa} + \Sigma}\right)G_{n\kappa}(r) = 0, \tag{7}$$

where $\Delta(r)$ denotes the difference potential, $\Delta(r) = V(r) - S(r)$, and $\Sigma(r)$ denotes the sum potential, $\Sigma(r) = V(r) + S(r)$. For the bound state solutions, one need the radial components satisfying the regularity conditions: $F_{n\kappa}(0) = G_{n\kappa}(0) = 0$ and $F_{n\kappa}(\infty) = G_{n\kappa}(\infty) = 0$.

3 Bound State Solutions of the Hyperbolic Potential

3.1 Spin symmetry solutions of the hyperbolic potential

Under the condition of the spin symmetry, i.e., $\Delta(r) = C = \text{const.}$, (6) turns to the following form

$$\left(-\frac{d^2}{dr^2} + \frac{\kappa(\kappa+1)}{r^2} + (M + E_{n\kappa} - C)\Sigma(r)\right) F_{n\kappa}(r) = (E_{n\kappa}^2 - M^2 + C(M - E_{n\kappa}))F_{n\kappa}(r). \tag{8}$$

We take the diatomic molecular hyperbolic potential model (1) as the sum potential $\Sigma(r)$, i.e.,

$$\Sigma(r) = D(1 - \sigma \coth \alpha r)^2. \tag{9}$$

For a diatomic molecular model instead of the nuclear model, we consider the reduced mass definition. If the nuclei have masses m_1 and m_2, the reduced mass is defined as $\mu = m_1 m_2/(m_1 + m_2)$ and in this point the diatomic molecular model can be included to the spin symmetry and pseudospin symmetry concept [36]. Substituting (9) into (8) leads us to obtain a Schrödinger-like equation for the upper component $F_{n\kappa}(r)$,

$$\begin{aligned}\left[-\frac{d^2}{dr^2} + D(M + E_{n\kappa} - C)(1 - \sigma \coth \alpha r)^2 + \frac{\kappa(\kappa+1)}{r^2}\right] F_{n\kappa}(r) \\ = (E_{n\kappa}^2 - M^2 + C(M - E_{n\kappa}))F_{n\kappa}(r).\end{aligned} \tag{10}$$

This equation can be solved analytically only for the s-wave ($\kappa = -1$) due to the centrifugal term $\kappa(\kappa+1)/r^2$. When $\alpha r \ll 1$, we use a new improved approximation scheme [31–35] to deal with the centrifugal term,

$$\frac{1}{r^2} \approx 4\alpha^2 \left[c_0 + \frac{e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2}\right], \tag{11}$$

where the parameter $c_0 = 1/12$ is a dimensionless constant, which is obtained by using the following power series

$$4\alpha^2\left(c_0 + \frac{e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2}\right) = 4\alpha^2\left[c_0 + \frac{1}{(2\alpha r)^2} - \frac{1}{12} + \frac{(2\alpha r)^2}{240} - \frac{(2\alpha r)^4}{6048} + O((2\alpha r)^6)\right]. \tag{12}$$

Substituting approximation expression (11) into (10), we have the following equation for the upper component $F_{n\kappa}(r)$,

$$\left(-\frac{d^2}{dr^2} + \frac{4\sigma\xi_1 + 4\alpha^2\kappa(\kappa+1)}{(1 - e^{-2\alpha r})^2} e^{-2\alpha r} - \frac{4\xi_1}{1 - e^{-2\alpha r}} e^{-2\alpha r}\right) F_{n\kappa}(r) = \tilde{E}_{n\kappa} F_{n\kappa}(r), \tag{13}$$

where ξ_1 and $\tilde{E}_{n\kappa}$ are defined as $\xi_1 = D\sigma(M + E_{n\kappa} - C)$ and $\tilde{E}_{n\kappa} = E^2_{n\kappa} - M^2 + C(M - E_{n\kappa}) - D(1-\sigma)^2(M + E_{n\kappa} - C) - 4\alpha^2\kappa(\kappa+1)c_0$, respectively. In terms of the basic concept of the supersymmetric shape invariance formalism [37, 38], we solve (13). Writing the ground-state upper component $F_{0,\kappa}(r)$ as follows

$$F_{0,\kappa}(r) = \exp\left(-\int W(r)dr\right), \tag{14}$$

where $W(r)$ is called a superpotential in supersymmetric quantum mechanics [37], and substituting it into (13) leads us to obtain the following equation for $W(r)$,

$$W^2(r) - \frac{dW(r)}{dr} = \frac{4\sigma\xi_1 + 4\alpha^2\kappa(\kappa+1)}{(1-e^{-2\alpha r})^2}e^{-2\alpha r} - \frac{4\xi_1}{1-e^{-2\alpha r}}e^{-2\alpha r} - \tilde{E}_{0,\kappa}, \tag{15}$$

where $\tilde{E}_{0,\kappa}$ is the ground-state energy. In order to make the superpotential $W(r)$ be compatible with the property of the right hand side of (15), we write the superpotential $W(r)$ as follows

$$W(r) = Q_1 + \frac{Q_2}{1-e^{-2\alpha r}}e^{-2\alpha r}, \tag{16}$$

where Q_1 and Q_2 are two constants. In terms of the superpotential function given in (16), we construct the following a pair of partner potentials

$$U_-(r) = W^2(r) - \frac{dW(r)}{dr^2} = Q_1^2 + \frac{(2Q_1Q_2 - Q_2^2)e^{-2\alpha r}}{1-e^{-2\alpha r}} + \frac{(Q_2^2 + 2\alpha Q_2)e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}, \tag{17}$$

$$U_+(r) = W^2(r) + \frac{dW(r)}{dr^2} = Q_1^2 + \frac{(2Q_1Q_2 - Q_2^2)e^{-2\alpha r}}{1-e^{-2\alpha r}} + \frac{(Q_2^2 - 2\alpha Q_2)e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}. \tag{18}$$

By comparing (17) with (15), we obtain the following three relationships

$$Q_1^2 = -\tilde{E}_{0,\kappa}, \tag{19}$$

$$2Q_1Q_2 - Q_2^2 = -4\xi_1, \tag{20}$$

$$Q_2^2 + 2\alpha Q_2 = 4\sigma\xi_1 + 4\alpha^2\kappa(\kappa+1). \tag{21}$$

Substituting (16) into (14) leads us to get the ground-state upper component $F_{0,\kappa}(r)$ as follows

$$F_{0,\kappa}(r) = e^{-Q_1 r}(1-e^{-2\alpha r})^{-\frac{Q_2}{2\alpha}}. \tag{22}$$

We consider the bound state solutions, which demand the upper component $F_{n\kappa}(r)$ satisfying the regularity conditions: $F_{n\kappa}(0) = F_{n\kappa}(\infty) = 0$. These boundary conditions allow us to obtain the restriction conditions: $Q_1 > 0$ and $Q_2 < 0$. Taking account of these restriction conditions and solving (20) and (21), we have

$$Q_1 = \frac{Q_2}{2} - \frac{2\xi_1}{Q_2}, \tag{23}$$

$$Q_2 = -\alpha\left(1 - \sqrt{(1+2\kappa)^2 + \frac{4\sigma\xi_1}{\alpha^2}}\right). \tag{24}$$

With the help of (20) and (23), we write the two supersymmetric partner potentials $U_-(r)$ and $U_+(r)$ as follows

$$U_-(r)=\left(\frac{Q_2}{2}-\frac{2\xi_1}{Q_2}\right)^2-\frac{4\xi_1}{1-e^{-2\alpha r}}e^{-2\alpha r}+\frac{Q_2^2+2\alpha Q_2}{(1-e^{-2\alpha r})^2}e^{-2\alpha r}, \tag{25}$$

$$U_+(r)=\left(\frac{Q_2}{2}-\frac{2\xi_1}{Q_2}\right)^2-\frac{4\xi_1}{1-e^{-2\alpha r}}e^{-2\alpha r}+\frac{Q_2^2-2\alpha Q_2}{(1-e^{-2\alpha r})^2}e^{-2\alpha r}. \tag{26}$$

By observing the above two equations, we get the following shape invariance relationship between the partner potentials $U_-(r)$ and $U_+(r)$,

$$U_+(r,a_0)=U_-(r,a_1)+R(a_1), \tag{27}$$

where $a_0=Q_2$, a_1 is a function of a_0, i.e., $a_1=h(a_0)=a_0-2\alpha$, and the reminder $R(a_1)$ is independent of r, $R(a_1)=(\frac{a_0}{2}-\frac{2\xi_1}{a_0})^2-(\frac{a_1}{2}-\frac{2\xi_1}{a_1})^2$. By using the shape invariance approach [38], the energy eigenvalues of the shape-invariant potential $U_-(r)$ can be exactly calculated and given by

$$\tilde{E}^{(-)}_{0,\kappa}=0, \tag{28}$$

$$\begin{aligned}
\tilde{E}^{(-)}_{n\kappa}&=\sum_{k=1}^{n}R(a_k)\\
&=R(a_1)+R(a_2)+\cdots+R(a_n)\\
&=\left(\frac{a_0}{2}-\frac{2\xi_1}{a_0}\right)^2-\left(\frac{a_1}{2}-\frac{2\xi_1}{a_1}\right)^2+\left(\frac{a_1}{2}-\frac{2\xi_1}{a_1}\right)^2-\left(\frac{a_2}{2}-\frac{2\xi_1}{a_2}\right)^2\\
&\quad+\cdots+\left(\frac{a_{n-1}}{2}-\frac{2\xi_1}{a_{n-1}}\right)^2-\left(\frac{a_n}{2}-\frac{2\xi_1}{a_n}\right)^2\\
&=\left(\frac{a_0}{2}-\frac{2\xi_1}{a_0}\right)^2-\left(\frac{a_n}{2}-\frac{2\xi_1}{a_n}\right)^2\\
&=\left(\frac{Q_2}{2}-\frac{2\xi_1}{Q_2}\right)^2-\left(\frac{Q_2-2n\alpha}{2}-\frac{2\xi_1}{Q_2-2n\alpha}\right)^2,
\end{aligned} \tag{29}$$

where the quantum number $n=0,1,2,\ldots$. From (13), (21) and (25), we find that the solution of $\tilde{E}_{n\kappa}$ in (13) can be given by

$$\tilde{E}_{n\kappa}=\tilde{E}^{(-)}_{n\kappa}+\tilde{E}_{0,\kappa}=\tilde{E}^{(-)}_{n\kappa}-\left(\frac{Q_2}{2}-\frac{2\xi_1}{Q_2}\right)^2. \tag{30}$$

Substituting (29) into (30) and using (24) allow us to obtain

$$\tilde{E}_{n\kappa}=-4\alpha^2\left(\frac{\xi_1/\alpha^2}{2n+1+\sqrt{(1+2\kappa)^2+4\sigma\xi_1/\alpha^2}}-\frac{2n+1+\sqrt{(1+2\kappa)^2+4\sigma\xi_1/\alpha^2}}{4}\right)^2. \tag{31}$$

Considering $\tilde{E}_{n\kappa}=E^2_{n\kappa}-M^2+C(M-E_{n\kappa})-D(1-\sigma)^2(M+E_{n\kappa}-C)-4\alpha^2\kappa(\kappa+1)c_0$ in (31), we can obtain the energy equation for the relativistic hyperbolic potential (9) with the spin symmetry in the Dirac theory,

$$\begin{aligned} &E_{n\kappa}^2 - M^2 + C(M - E_{n\kappa}) \\ &\quad = 4\alpha^2\kappa(\kappa+1)c_0 + D(1-\sigma)^2(M + E_{n\kappa} - C) \\ &\qquad - 4\alpha^2\left(\frac{\xi_1/\alpha^2}{2n+1+\sqrt{(1+2\kappa)^2+4\sigma\xi_1/\alpha^2}} - \frac{2n+1+\sqrt{(1+2\kappa)^2+4\sigma\xi_1/\alpha^2}}{4}\right)^2, \end{aligned} \tag{32}$$

where the quantum number $n = 0, 1, 2, \ldots$.

By employing the function analysis method, we determine the unnormalized excited state wave functions. Substituting (31) into (13) leads us to write (13) as follows

$$\begin{aligned} &\left(-\frac{d^2}{dr^2} + \frac{4\sigma\xi_1 + 4\alpha^2\kappa(\kappa+1)}{(1-e^{-2\alpha r})^2}e^{-2\alpha r} - \frac{4\xi_1}{1-e^{-2\alpha r}}e^{-2\alpha r}\right)F_{n\kappa}(r) \\ &\quad = -4\alpha^2\left(\frac{\xi_1/\alpha^2}{2n+1+\sqrt{(1+2\kappa)^2+4\sigma\xi_1/\alpha^2}}\right. \\ &\qquad \left. - \frac{2n+1+\sqrt{(1+2\kappa)^2+4\sigma\xi_1/\alpha^2}}{4}\right)^2 F_{n\kappa}(r). \end{aligned} \tag{33}$$

Introducing a new variable $z = e^{-2\alpha r}$ and writing the radial wave function $F_{n\kappa}(r)$ as $F_{n\kappa}(z) = (1-z)^{\delta_1} z^{\eta_1} f_{n\kappa}(z)$, (33) can be further written into the following form

$$(1-z)z\frac{d^2 f_{n\kappa}(z)}{dz^2} + [1+2\eta_1 - (1+2\delta_1+2\eta_1)z]\frac{df_{n\kappa}(z)}{dz} - (-n)(n+2\delta_1+2\eta_1)f_{n\kappa}(z) = 0, \tag{34}$$

where the parameters δ_1 and η_1 are defined as

$$\delta_1 = \frac{1+\sqrt{(1+2\kappa)^2+4\sigma\xi_1/\alpha^2}}{2}, \tag{35}$$

$$\eta_1 = \frac{\xi_1/\alpha^2}{2n+1+\sqrt{(1+2\kappa)^2+4\sigma\xi_1/\alpha^2}} - \frac{2n+1+\sqrt{(1+2\kappa)^2+4\sigma\xi_1/\alpha^2}}{4}. \tag{36}$$

Equation (34) is the well-known differential equation satisfied by the hypergeometric function ${}_2F_1(-n, n+2\delta_1+2\eta_1; 1+2\eta_1; z)$, i.e.,

$$\begin{aligned} f_{n\kappa}(z) &= {}_2F_1(-n, n+2\delta_1+2\eta_1; 1+2\eta_1; z) \\ &= \frac{\Gamma(1+2\eta_1)}{\Gamma(-n)\Gamma(n+2\delta_1+2\eta_1)}\sum_{k=0}^{\infty}\frac{\Gamma(-n+k)\Gamma(n+2\delta_1+2\eta_1+k)}{\Gamma(1+2\eta_1+k)}\frac{z^k}{k!}. \end{aligned} \tag{37}$$

Replacing the function $f_{n\kappa}(z)$ with the hypergeometric function ${}_2F_1(-n, n+2\delta_1+2\eta_1; 1+2\eta_1; z)$ and changing the new variable z into the original variable r, we obtain the unnormalized radial wave function $F_{n\kappa}(r)$ as follows

$$F_{n\kappa}(r) = (1-e^{-2\alpha r})^{\delta_1} e^{-2\eta_1 \alpha r}\, {}_2F_1(-n, n+2\delta_1+2\eta_1; 1+2\eta_1; e^{-2\alpha r}). \tag{38}$$

Substituting $F_{n\kappa}(r)$ given in (38) into (4), we obtain the lower spinor component $G_{n\kappa}(r)$ corresponding to the energy level $E_{n\kappa}$,

$$G_{n\kappa}(r) = \frac{1}{M + E_{n\kappa} - C}\left(\frac{2\alpha\delta_1 e^{-2\alpha r}}{1 - e^{-2\alpha r}} - 2\alpha\eta_1 + \frac{\kappa}{r}\right)F_{n\kappa}(r) + \frac{2n\alpha(n + 2\delta_1 + 2\eta_1)}{(M + E_{n\kappa} - C)(1 + 2\eta_1)}(1 - e^{-2\alpha r})^{\delta_1}(e^{-2\alpha r})^{\eta_1+1} \times {}_2F_1(1 - n, n + 1 + 2\delta_1 + 2\eta_1; 2 + 2\eta_1; e^{-2\alpha r}). \quad (39)$$

It should be pointed out that the hypergeometric function ${}_2F_1(1 - n, n + 1 + 2\delta_1 + 2\eta_1; 2 + 2\eta_1; e^{-2\alpha r})$ does not terminate for $n = 0$ and it does not diverge for all values of real parameters δ_1 and η_1. Employing $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ into (3), we can get the spinor wave functions for the relativistic hyperbolic potential (9) with the spin symmetry in the Dirac theory. Equations (38) and (39) tell us that the upper component $F_{n\kappa}(r)$ and lower component $G_{n\kappa}(r)$ can satisfy the boundary conditions for the bound states when $\delta_1 > 1$ and $\eta_1 > 0$. Equation (39) also show that only bound positive energy states exist in the presence of the spin symmetry, otherwise the lower spinor component $G_{n\kappa}(r)$ will diverge when $E_{n\kappa} = -M$ and $C = 0$. The energy level $E_{n\kappa}$ is defined implicitly by energy equation (32) which is a rather complicated transcendental equation having many solutions for given values of n and κ. For these solutions, we can only choose a suitable one, which can make the upper spinor component $F_{n\kappa}(r)$ and lower spinor component $G_{n\kappa}(r)$ satisfy the restriction conditions for the bound states.

When the parameter α in approximation (11) goes to zero, the approximation (11) to the centrifugal term becomes into $1/r^2$, i.e.,

$$\lim_{\alpha\to 0}\left[4\alpha^2\left(c_0 + \frac{e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2}\right)\right] = \frac{1}{r^2}. \quad (40)$$

Equation (40) tells us that the usual centrifugal term is the limit of the approximation (11) to the centrifugal term when α goes to zero. In the limit of $\alpha \to 0$, the limits of the energy eigenvalues and Dirac spinor components turn to the following forms in the presence of the spin symmetry,

$$\lim_{\alpha\to 0} E_{n\kappa} = M, \qquad \lim_{\alpha\to 0} F_{n\kappa}(r) = 0, \qquad \lim_{\alpha\to 0} G_{n\kappa}(r) = 0. \quad (41)$$

These results tell us that the lower component $G_{n\kappa}(r)$ and upper component $F_{n\kappa}(r)$ become unbound when $\alpha \to 0$ and the eigenenergies become a constant. In fact, the limit of the sum potential $\Sigma(r)$ given in (9) become infinity in the limit of $\alpha \to 0$, that is

$$\lim_{\alpha\to 0} \Sigma(r) = \infty. \quad (42)$$

This shows that when α goes to zero a Dirac particle could not be trapped by the relativistic hyperbolic potential (9), which does not possess any bound state under the condition of the spin symmetry.

3.2 Pseudospin symmetry solutions of the hyperbolic potential

Under the condition of the pseudospin symmetry, i.e., $\Sigma(r) = C = \text{const.}$, (7) turns to the following form

$$\left(-\frac{d^2}{dr^2} - (M - E_{n\kappa} + C)\Delta(r) + \frac{\kappa(\kappa-1)}{r^2}\right)G_{n\kappa}(r) = (E_{n\kappa}^2 - M^2 - C(M + E_{n\kappa}))G_{n\kappa}(r). \tag{43}$$

We choose the hyperbolic potential (1) as the difference potential $\Delta(r)$, that is

$$\Delta(r) = D(1 - \sigma \coth \alpha r)^2. \tag{44}$$

Substituting (44) and approximation (11) into (43), we have a Schrödinger-like equation for the lower component $G_{n\kappa}(r)$,

$$\left(-\frac{d^2}{dr^2} + \frac{4\sigma\xi_2 + 4\alpha^2\kappa(\kappa-1)}{(1-e^{-2\alpha r})^2}e^{-2\alpha r} - \frac{4\xi_2}{1-e^{-2\alpha r}}e^{-2\alpha r}\right)G_{n\kappa}(r) = \tilde{E}_{n\kappa}G_{n\kappa}(r), \tag{45}$$

where ξ_2 and $\tilde{E}_{n\kappa}$ are defined as $\xi_2 = -D\sigma(M - E_{n\kappa} + C)$ and $\tilde{E}_{n\kappa} = E_{n\kappa}^2 - M^2 - C(M + E_{n\kappa}) + D(1-\sigma^2)(M - E_{n\kappa} + C) - 4\alpha^2\kappa(\kappa-1)c_0$, respectively. By employing the same procedure of solving (13), we solve (45) and obtain the energy equation for the relativistic hyperbolic potential (44) with the pseudospin symmetry in the Dirac theory,

$$\begin{aligned} E_{n\kappa}^2 - M^2 - C(M + E_{n\kappa}) &= 4\alpha^2\kappa(\kappa-1)c_0 - D(1-\sigma)^2(M - E_{n\kappa} + C) \\ &\quad - 4\alpha^2\left(\frac{\xi_{21}/\alpha^2}{2n+1+\sqrt{(1-2\kappa)^2 + 4\sigma\xi_2/\alpha^2}}\right. \\ &\quad \left. - \frac{2n+1+\sqrt{(1-2\kappa)^2+4\sigma\xi_2/\alpha^2}}{4}\right)^2, \end{aligned} \tag{46}$$

where the quantum number $n = 0, 1, 2, \ldots$. The unnormalized lower radial component is given by

$$G_{n\kappa}(r) = (1 - e^{-2\alpha r})^{\delta_2} e^{-2\eta_2 \alpha r}\, {}_2F_1(-n, n + 2\delta_2 + 2\eta_2; 1 + 2\eta_2; e^{-2\alpha r}), \tag{47}$$

where δ_2 and η_2 are defined as

$$\delta_2 = \frac{1 + \sqrt{(1-2\kappa)^2 + 4\sigma\xi_2/\alpha^2}}{2}, \tag{48}$$

$$\eta_2 = \frac{\xi_2/\alpha^2}{2n+1+\sqrt{(1-2\kappa)^2+4\sigma\xi_2/\alpha^2}} - \frac{2n+1+\sqrt{(1-2\kappa)^2+4\sigma\xi_2/\alpha^2}}{4}. \tag{49}$$

Substituting the expression of $G_{n\kappa}(r)$ given in (47) into (5), we obtain the upper spinor component $F_{n\kappa}(r)$ corresponding to the energy level $E_{n\kappa}$,

$$\begin{aligned} F_{n\kappa}(r) = &\frac{1}{M - E_{n\kappa} + C}\left(\frac{2\alpha\delta_2 e^{-2\alpha r}}{1 - e^{-2\alpha r}} - 2\alpha\eta_2 - \frac{\kappa}{r}\right)F_{n\kappa}(r) \\ &+ \frac{2n\alpha(n + 2\delta_2 + 2\eta_2)}{(M - E_{n\kappa} + C)(1 + 2\eta_2)}(1 - e^{-2\alpha r})^{\delta_2}(e^{-2\alpha r})^{\eta_2+1} \\ &\times {}_2F_1(1 - n, n + 1 + 2\delta_2 + 2\eta_2; 2 + 2\eta_2; e^{-2\alpha r}). \end{aligned} \tag{50}$$

After choosing a set of the values of n and κ, we can obtain many solutions for the energy level $E_{n\kappa}$ from (46), however, we can only choose suitable one that can make the lower spinor component $G_{n\kappa}(r)$ and upper spinor component $F_{n\kappa}(r)$ satisfy the restriction conditions: $\delta_2 > 1$ and $\eta_2 > 0$. Equation (50) shows that only bound negative energy states exist under the condition of the pseudospin symmetry, otherwise the upper spinor component $F_{n\kappa}(r)$ will diverge if $E_{n\kappa} = M$ and $C = 0$.

Equation (11) shows that the usual pseudo-centrifugal term is the limit of the approximation (11) to the pseudo-centrifugal term when α goes to zero. In the limit of $\alpha \to 0$, the limits of the energy eigenvalues and Dirac spinor components turn to the following forms in the presence of the pseudospin symmetry,

$$\lim_{\alpha\to 0} E_{n\kappa} = -M, \qquad \lim_{\alpha\to 0} G_{n\kappa}(r) = 0, \qquad \lim_{\alpha\to 0} F_{n\kappa}(r) = 0. \tag{51}$$

Obviously, when $\alpha \to 0$, the lower component $G_{n\kappa}(r)$ and upper component $F_{n\kappa}(r)$ become unbound and the eigenenergies become a constant. This is due to the fact that when the limit of α goes to zero, the limit of the difference potential $\Delta(r)$ given in (44) goes to infinity, i.e., $\lim_{\alpha\to 0} \Delta(r) = \infty$, the relativistic hyperbolic potential (44) could not trap a Dirac particle under the condition of the pseudospin symmetry.

4 Conclusions

In this work, we have approximately investigated the bound state solutions of the Dirac equation with the hyperbolic potential (1) in the presence of the spin symmetry and pseudospin symmetry. By using a new improved approximation scheme to deal with the centrifugal term and pseudo-centrifugal term, we have solved approximately the Dirac equations with the relativistic hyperbolic potentials (9) and (44) in terms of the basic concept of the supersymmetric shape invariance formalism and the function analysis method. The energy eigenvalue equations and associated two-component spinors of the relativistic hyperbolic potentials (9) and (44) have been approximately obtained for the arbitrary spin-orbit quantum number κ.

Acknowledgements This work was supported by the National Natural Science Foundation of China under Grant No. 10675097, the Natural Science Foundation of the Sichuan Education Bureau of China under Grant No. 07ZA140 and the Science and Technology Foundation of Southwest Petroleum University under Grant No. 2007XJZ073.

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